THE ARKHIPOV-BEZRUKAVNIKOV EQUIVALENCE

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Abstract. These expository notes accompany my BunG seminar talk on the Arkhipov-Bezrukavnikov equivalence. First, we will setup the machinery of central sheaves and Wakimoto sheaves. Then, we will explain the geometry of coherent sheaves on the Springer resolution. We will use this to build a functor from coherent sheaves on the Springer resolution to the antispherical Hecke category. Finally, we will relate the antispherical Hecke category with the Iwahori-Whittaker category.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{F}$ where $\mathbb{F} = \overline{\mathbb{F}_p}$. Let B denote a Borel subgroup of G with maximal torus T. Let $W_{\text{ext}} = W \ltimes X^\vee$ denote the extended affine Weyl group. The *antispherical module* M^{asph} is the right $\mathbb{Z}[W_{\text{ext}}]$-module given by the tensor product

$$M^{\text{asph}} := \mathbb{Z} \otimes_{\mathbb{Z}[W]} \mathbb{Z}[W_{\text{ext}}] \tag{1}$$

where $\mathbb{Z}[W]$ acts on $\mathbb{Z}$ by the sign representation. The antispherical module has two other useful descriptions. The first is provided by the K-theory of $G^\vee$-equivariant coherent sheaves on the Springer resolution $\tilde{\mathcal{N}}$,

$$[D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}})] \cong M^{\text{asph}}. \tag{2}$$

The second is provided by q-deforming M^{asph} into the Whittaker modules M_q^{asph} in the theory of p-adic groups. The Whittaker modules are described by "Iwahori-Whittaker" functions on $G_\circ(\mathbb{F}_q((x)))$ for $G = \mathbb{F} \otimes_{\mathbb{F}_q} G_\circ$. These three descriptions of the antispherical module give rise to three possible categorifications:

(1) P_I^{asph}, an abelian category constructed by taking a quotient of perverse sheaves on the affine flag variety (corresponds to 1);
(2) $D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}})$, the derived $G^\vee$-equivariant coherent sheaves on $\tilde{\mathcal{N}}$ (corresponds to 2);
(3) $P_{\mathcal{IW}}$, an abelian category of Iwahori-Whittaker perverse sheaves given by a version of the sheaf-function dictionary for Iwahori-Whittaker functions.

The celebrated equivalences of Arkhipov and Bezrukavnikov upgrade the different descriptions of the antispherical module to equivalences of categories.

Theorem 1.0.1 (Arkhipov-Bezrukavnikov, [ArBe])**.** *There are equivalences of triangulated categories*

$$D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \cong D^b P_I^{\text{asph}} \cong D^b P_{\mathcal{IW}}.$$

Outline 1.0.2. *(1) We will discuss central sheaves and Wakimoto sheaves. These form an important class of sheaves on the affine flag variety, and* *will be crucial for constructing the functor from coherent sheaves to constructible sheaves.*

(2) *We will then study the geometry of the Springer resolution and use central sheaves and Wakimoto sheaves to define the functor* $D^b\operatorname{Coh}(\tilde{\mathcal{N}}/G^\vee) \to D^b P_I^{\mathrm{asph}}$. *This is in some sense, the hardest part.*
(3) *Next, we will explain the construction of Iwahori-Whittaker perverse sheaves, as well as the functor* $P_I^{\mathrm{asph}} \to P_{\mathcal{IW}}$.
(4) *After constructing the functor* $D^b\operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \to D^b P_{\mathcal{IW}}$, *the proof that it is an equivalence is rather simple.*

2. Central Sheaves and Wakimoto Sheaves

2.1. **p-Adic motivation.** Let G be a connected reductive group over a field $\mathbb{F}_q$ and let G_F be the corresponding group over $F = \mathbb{F}_q((t))$. Let $G_{\mathcal{O}}$ be the maximal compact subgroup of G_F (here $\mathcal{O} = \mathbb{F}_q[[t]]$). Let $I \subset G_{\mathcal{O}}$ be the Iwahori subgroup.

Consider the map

$$\pi : \mathcal{C}(I\backslash G_F/I) \to \mathcal{C}(G_{\mathcal{O}}\backslash G_F/I)^1$$
$$f \mapsto \int_{G_{\mathcal{O}}} f(x\cdot g)dx$$

It is easy to check that π restricts to an algebra map

$$Z(\mathcal{C}(I\backslash G_F/I)) \to \mathcal{C}(G_{\mathcal{O}}\backslash G_F/G_{\mathcal{O}})$$

Moreover, by a theorem of Bernstein, the latter map is in fact an isomorphism of algebras.

The basic idea of central sheaves is to provide a geometric categorification of the inverse of this isomorphism. In order to construct this functor, one needs a theory of nearby cycles.

2.2. **Nearby cycles.** Let X be a complex algebraic variety, and let $f : X \to \mathbb{C}$ be an algebraic map. We consider the subvarieties

$$X^\times := f^{-1}(\mathbb{C}^\times), \qquad X^0 := f^{-1}(0).$$

We can also consider the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$, and set $\tilde{X}^\times = X \times_{\mathbb{C}^\times} \mathbb{C}$.[2] We have the following commutative diagram

$$\begin{tikzcd} X^0 \arrow[r, hook, "i"] \arrow[d, "f"] & X \arrow[d, "f"] & X^\times \arrow[l, hook', "j"'] \arrow[d, "f"] & \tilde{X}^\times \arrow[l, "\exp_X"'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^\times \arrow[l, hook'] & \mathbb{C} \arrow[l, "\exp"] \end{tikzcd} \tag{3}$$

Definition 2.2.1. *The **nearby cycles functor** is the functor*[3]

$$\Psi_f : D^b_{\mathrm{c}}(X^\times) \to D^b_{\mathrm{c}}(X^0)$$
$$\mathcal{F} \mapsto (i^* j_* \exp_{X*} \exp_X^* \mathcal{F})[-1]$$

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[1] The notation here means compactly supported invariant functions. The domain of π is the *affine Hecke algebra* and the codomain of π is *spherical Hecke algebra.*

[2] $\tilde{X}^\times$ is an analytic space, but not an algebraic variety.

[3] For now, one should consider these as sheaves with $\Bbbk$-coefficients where $\Bbbk$ is an algebraic closed field of characteristic 0.

The theory of nearby cycles is rich and very general. It is often useful to black box the geometry of the functor and instead think about its formal sheaf theoretic properties.

Proposition 2.2.2. *(1) The functor Ψ_f sends bounded constructible complexes to bounded constructible complexes.*[4]
(2) The functor Ψ_f is perverse t-exact.
(3) If $\phi : Y \to X$ is a smooth map, then ϕ^* commutes with nearby cycles, i.e., there is an isomorphism of functors

$$\Psi_{f\circ\phi} \circ (\phi\mid_{(f\circ\phi)^{-1}(\mathbb{C}^\times)})^* \cong (\phi\mid_{(f\circ\phi)^{-1}(\{0\})})^* \circ \Psi_f.$$

(4) If $\phi : Y \to X$ is a proper map, then there is an isomorphism of functors

$$\Psi_f \circ (\phi\mid_{(f\circ\phi)^{-1}(\mathbb{C}^\times)})_* = (\phi\mid_{(f\circ\phi)^{-1}(\{0\})})_* \Psi_{f\circ\phi}.$$

(5) If $f : X \times \mathbb{C} \to \mathbb{C}$ is the projection map, and $\mathcal{F} \in D^b_{\mathrm{c}}(X)$, then there is a natural isomorphism of sheaves

$$\Psi_f(\mathcal{F} \boxtimes \underline{\Bbbk}_{\mathbb{C}^\times}[1]) \cong \mathcal{F}.$$

2.3. **Affine Grassmannian and affine flag variety.** Let G be a connected reductive group over $\mathbb{C}$. Consider its loop group LG and its positive loop group L^+G. Fix a Borel $B^+ = T \ltimes U$ with opposite Borel $B^- = T \ltimes U^-$. We let I denote the Iwahori subgroup of L^+G corresponding to B^-. Let Gr_G denote the affine Grassmannian of G, explicitly, it is the fppf-sheafification of the quotient LG/L^+G. Similarly, let Fl_G denote the affine flag variety of G, explicitly, it is the fppf-sheafification of the quotient LG/I.

2.4. **Gaitsgory's central functor.** The construction of the central functor requires the introduction of a new geometric object, sometimes called the *central affine Grassmannian*. It is an ind-scheme $\mathrm{Gr}_G^{\mathrm{Cen}}$ equipped with a regular map $f : \mathrm{Gr}_G^{\mathrm{Cen}} \to \mathbb{C}$. The central affine Grassmannian satisfies

$$\mathrm{Gr}_G^{\mathrm{Cen},\times} := f^{-1}(\mathbb{C}^\times) = \mathrm{Gr}_G \times \mathbb{C}^\times, \qquad \mathrm{Gr}_G^{\mathrm{Cen},\times} := f^{-1}(\mathbb{C}^\times) = \mathrm{Fl}_G\,.$$

In this way, the central affine Grassmannian can be thought of as providing a deformation of the affine Grassmannian to the affine flag variety.

We are now in the situation where we can apply the nearby cycles construction of the previous section.[5] Namely, we define a functor

$$Z : D^b_{\mathrm{c}}(\mathrm{Gr}_G) \to D^b_{\mathrm{c}}(\mathrm{Fl}_G)$$
$$\mathcal{F} \mapsto \Psi_f(\mathcal{F} \boxtimes \underline{\Bbbk}_{\mathbb{C}^\times}[1]).$$

This can be upgraded to a functor of equivariant sheaves[6]

$$Z : D^b_{\mathrm{c}}(L^+G\backslash \mathrm{Gr}_G) \to D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G).$$

We call this functor *Gaitsgory's central functor*.

[4]This is not obvious since $\exp_X$ is not an algebraic map.

[5]Technically, our construction of nearby cycles is for complex algebraic varieties and not ind-varieties. Nonetheless, the central affine Grassmannian can be constructed as an inductive limit of finite dimensional varieties over $\mathbb{A}^1$ whose fibers generically look like finite dimensional approximations of Gr_G and at 0 looks like finite dimensional approximations of Fl_G.

[6]This requires an additional component where we construct a group ind-scheme $L^+\mathcal{G}$ over $\mathbb{A}^1$ whose fibers generically are L^+G except at 0 which is I. See Sections 2.2 and 10.3 of [AcRi].

Recall that $D^b_{\mathrm{c}}(L^+G\backslash \mathrm{Gr}_G)$ and $D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G)$ can be equipped with monoidal structures, denoted $\star^{L^+G}$ and $\star^I$, respectively.[7]

Theorem 2.4.1 (Gaitsgory [Gai1])**.** *(1) The functor Z is monoidal.*
(2) The functor Z is perverse t-exact.
(3) The functor Z produces objects in the center of $D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G)$, i.e., for $\mathcal{F} \in D^b_{\mathrm{c}}(L^+G\backslash \mathrm{Gr}_G)$ and $\mathcal{G} \in D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G)$, there are canonical isomorphsisms
$$Z(\mathcal{F}) \star^I \mathcal{G} \cong \mathcal{G} \star^I Z(\mathcal{F}).$$
(4) For $\mathcal{F} \in \mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G)$ and $\mathcal{G} \in \mathrm{Perv}(I\backslash \mathrm{Fl})$, the convolution $Z(\mathcal{F}) \star \mathcal{G}$ is perverse.
(5) For any $\mathcal{F} \in D^b_{\mathrm{c}}(L^+G\backslash \mathrm{Gr}_G)$, there is a canonical isomorphism $\pi_* \circ Z(\mathcal{F}) \cong \mathcal{F}$ for canonical map $\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$.

Proof. We will prove (2) and (5) to highlight how these statements mostly follow from Proposition 2.2.2.

The proof of (2) is nearly obvious. Namely, the functor $\mathcal{F} \mapsto \mathcal{F} \boxtimes \underline{\Bbbk}_{\mathbb{C}}[1]$ is perverse t-exact and by Proposition 2.2.2 (2), nearby cycles is perverse t-exact.

To prove (5), one first constructs a map $\varpi : \mathrm{Gr}^{\mathrm{Cen}}_G \to \mathrm{Gr}_G \times \mathbb{C}$ such that $\varpi|_{\mathrm{Gr}_G \times \mathbb{C}^\times} = \mathrm{id}_{\mathrm{Gr}_G \times \mathbb{C}^\times}$, $\varpi|_{\mathrm{Gr}_G \times \{0\}} = \pi$, and $\mathrm{pr}_2 \circ \varpi = f$. By Proposition 2.2.2 (4) and Proposition 2.2.2 (5), there are isomorphism
$$\mathcal{F} \cong \Psi_{\mathrm{pr}_2}(\mathcal{F} \boxtimes \underline{\Bbbk}_{\mathbb{C}}[1]) \cong \pi_* \Psi_f(\mathcal{F} \boxtimes \underline{\Bbbk}_{\mathbb{C}}[1]) = \pi_* Z(\mathcal{F}).$$
$\square$

Motivated by Theorem 2.4.1, perverse sheaves of the form $Z(\mathcal{F})$, where $\mathcal{F} \in \mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G)$, are called *central sheaves*.

Aside 2.4.2. *In light of Theorem 2.4.1 and Bernstein's theorem for p-adic groups, one may be tempted to conjecture that there is an equivalence*
$$D^b_{\mathrm{c}}(L^+G\backslash \mathrm{Gr}_G) \cong Z(D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G))$$
induced by Gaitsgory's central functor. This is in fact not the case, the center of the affine Hecke category is in fact much larger. See for example [BNP] for a spectral description of the center.

Remark 2.4.3. *Recall the Geometric Satake equivalence gives an equivalence of tensor categories*
$$\mathrm{Sat} : \mathrm{Rep}(G^\vee) \xrightarrow{\sim} \mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G).$$
In this way, there is a central functor
$$\mathrm{Rep}(G^\vee) \to \mathrm{Perv}(I\backslash \mathrm{Fl}_G)$$
given by composing the Satake equivalence with Gaitsgory's central functor. We will often abuse notation and write Z for either functor.

[7]Convolution on Gr_G is perverse t-exact, convolution on Fl_G is not t-exact.

2.5. **Wakimoto Sheaves.** Recall the Bruhat decomposition stratifies the affine flag variety into subvarieties

$$\mathrm{Fl}_G = \bigsqcup_{w \in W_{\mathrm{ext}}} \mathrm{Fl}_{G,w}$$

Let $j_w : \mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$ denote the embedding. Each $\mathrm{Fl}_{G,w}$ is an affine space of dimension $\ell(w)$ where $\ell : W_{\mathrm{ext}} \to \mathbb{Z}_{\geq 0}$ is the length function for the quasi-Coxeter structure on W_{ext}. For each $w \in W_{\mathrm{ext}}$ we define perverse sheaves

$$\Delta_w = j_{w!}\underline{\Bbbk}_{\mathrm{Fl}_{G,w}}[\ell(w)], \qquad \nabla_w = j_{w*}\underline{\Bbbk}_{\mathrm{Fl}_{G,w}}[\ell(w)].$$

These perverse sheaves are called the *standards* and *costandard* sheaves, respectively. One can also consider the simple perverse sheaf, denoted IC_w, given by the image of the natural map $\Delta_w \to \nabla_w$. The standards, costandards, and IC-complexes are naturally I-equivariant.

The standard and costandard sheaves behave particularly nice under convolution.

Proposition 2.5.1. *(1) For any $x, y \in W_{\mathrm{ext}}$ such that $\ell(xy) = \ell(x) + \ell(y)$, there exists canonical isomorphisms*

$$\Delta_x \star^I \Delta_y \cong \Delta_{xy}, \qquad \nabla_x \star^I \nabla_y \cong \nabla_{xy}.$$

(2) For any $x \in W_{\mathrm{ext}}$, there exist canonical isomorphisms

$$\Delta_x \star^I \nabla_{x^{-1}} \cong \Delta_e \cong \nabla_{x^{-1}} \star^I \Delta_x.$$

(3) For any $x, y \in W_{\mathrm{ext}}$, the sheaf $\Delta_x \star^I \nabla_y$ is perverse.

For $\lambda \in X^\vee$, choose $\lambda_1, \lambda_2 \in X^\vee_+$ such that $\lambda = \lambda_1 - \lambda_2$. We define a perverse sheaf

$$\mathcal{W}_\lambda = \nabla_{\lambda_1} \star^I \Delta_{-\lambda_2}.$$

The collection of such sheaves are called *Wakimoto sheaves.*

Proposition 2.5.2 (Mirković). *Let $\lambda, \mu \in X^\vee$*

1. *The sheaf $\mathcal{W}_\lambda$ does not depend on the choice of $\lambda = \lambda_1 - \lambda_2$.*
2. *The sheaf $\mathcal{W}_\lambda$ is supported on $\overline{\mathrm{Fl}_{G,\lambda}}$.*
3. *We have that*

$$\mathcal{W}_\lambda \star^I \mathcal{W}_\mu \cong \mathcal{W}_{\lambda+\mu}.$$

Theorem 2.5.3 (Arkhipov-Bezrukavnikov, [ArBe]). *(1) For any $\mathcal{F} \in \mathrm{Perv}(L^+G \backslash \mathrm{Gr}_G)$, the perverse sheaf $Z(\mathcal{F})$ admits a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = Z(\mathcal{F})$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is of the form $\mathcal{W}_{\lambda_i}$ for some $\lambda_i \in X^\vee$.

(2) For any $\lambda \in X^\vee$, the multiplicity of $\mathcal{W}_\lambda$ in the above filtration of $Z(\mathcal{F})$ is the dimension of the λ-weight space of $\mathrm{Sat}^{-1}(\mathcal{F})$.

We will denote P_I^{Wak} for the full subcategory of $\mathrm{Perv}(I \backslash \mathrm{Fl}_G)$ consisting of perverse sheaves which admit Wakimoto filtrations.

Lemma 2.5.4. *The category P_I^{Wak} is stable under convolution in $D^b_c(I \backslash \mathrm{Fl}_G)$.*

2.6. **The Antispherical Quotient.** It will be useful to shorten our notation for I-equivariant perverse sheaves on Fl_G. Namely, we will write $P_I = \mathrm{Perv}(I\backslash \mathrm{Fl}_G)$.

Let ${}^fW_{\mathrm{ext}} \subseteq W_{\mathrm{ext}}$ be the subset of elements w which are minimal in Ww. We define the *antispherical Hecke category*, denoted P_I^{asph}, as the abelian category constructed by taking the Serre quotient of P_I by the Serre subcategory generated by simple objects IC_w with $w \notin {}^fW_{\mathrm{ext}}$. It comes equipped with a quotient functor

$$P_I \to P_I^{\mathrm{asph}}.$$

3. Coherent Sheaves on the Springer Resolution

3.1. **The Springer resolution.** Let $\mathcal{N}$ denote the nilpotent cone of $G^\vee$. We can consider the Springer resolution

$$\tilde{\mathcal{N}} := G^\vee \times^{B^\vee} \mathfrak{n}^\vee.$$

The Springer resolution comes equipped with a moment map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ along with a projection map $\tilde{\mathcal{N}} \to G^\vee/B^\vee$. The latter map makes $\tilde{\mathcal{N}}$ into a vector bundle over $G^\vee/B^\vee$. Moreover, it is a sub-vector bundle of $\mathfrak{g}^\vee \times G^\vee/B^\vee$. For each $\lambda \in X^\vee$, there is a line bundle $\mathcal{O}_{G^\vee/B^\vee}(\lambda)$ on $G^\vee/B^\vee$. We will write $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ for the pullback of $\mathcal{O}_{G^\vee/B^\vee}(\lambda)$ to $\tilde{\mathcal{N}}$.

Lemma 3.1.1. *The category $D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})$ is generated by the following classes of objects:*

(1) the line bundles $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$, for $\lambda \in X^\vee$;
(2) the objects of the form $V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ where $V \in \mathrm{Rep}(G^\vee)$ and $\lambda \in X^\vee_+$.

We will denote by $\hat{\mathcal{N}}$ the preimage of $\tilde{\mathcal{N}}$ under the map $\mathfrak{g}^\vee \times G^\vee/U^\vee \to G^\vee/B^\vee$. We will also consider $\mathcal{X} := \mathrm{Spec}(\mathcal{O}(G^\vee/U^\vee))$, the affine completion of $G^\vee/U^\vee$. Note that $G^\vee/U^\vee$ is open $\mathcal{X}$. We will denote $\partial\mathcal{X}$ for the complement of $G^\vee/U^\vee$ in $\mathcal{X}$. It can easily be checked that the infinitesimal universal stabilizer for the $G^\vee$-action on $G^\vee/U^\vee$ is $\hat{\mathcal{N}}$. We then define $\hat{\mathcal{N}}_{\mathcal{X}}$ as the infinitesimal universal stabilizer for the $G^\vee$-action on $\mathcal{X}$. It is a closed subscheme of $\mathfrak{g}^\vee \times \mathcal{X}$, and contains $\hat{\mathcal{N}}$ as an open subvariety. We summarize the above objects along with maps between them by the diagram below:

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[d, hook] & \hat{\mathcal{N}} \arrow[l, "T^\vee\text{-torsor}"'] \arrow[r, hook, "\text{open}"] \arrow[d, hook] & \hat{\mathcal{N}}_{\mathcal{X}} \arrow[d, hook] \\
\mathfrak{g}^\vee \times G^\vee/B^\vee & \mathfrak{g}^\vee \times G^\vee/U^\vee \arrow[l] \arrow[r] & \mathfrak{g}^\vee \times \mathcal{X}
\end{tikzcd} \tag{4}$$

The vertical maps are all inclusions of closed subschemes.

We will denote $\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}})$ for the full additive subcategory of $\mathrm{Coh}^{G^\vee \times T^\vee}(\hat{\mathcal{N}}_{\mathcal{X}})$ consisting of free coherent sheaves, i.e. those of the form $V \otimes \mathcal{O}_{\hat{\mathcal{N}}_{\mathcal{X}}}$ for $V \in \mathrm{Rep}(G^\vee \times T^\vee)$. We can further consider the full subcategory $K^b\,\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}})_{\partial\mathcal{X}}$ of $K^b\,\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}})$ consisting of objects whose cohomology is supported set-theoretically on the preimage of $\partial\mathcal{X}$ under the projection $\hat{\mathcal{N}}_{\mathcal{X}} \to \mathcal{X}$.

Proposition 3.1.2. *There is an functor*

$$K^b\,\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}}) \to D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})$$

which factors uniquely through an equivalence of triangulated categories

$$K^b \operatorname{Coh}^{G^\vee \times T^\vee}_{\text{free}}(\hat{\mathcal{N}}_{\mathcal{X}})/K^b \operatorname{Coh}^{G^\vee \times T^\vee}_{\text{free}}(\hat{\mathcal{N}}_{\mathcal{X}})_{\partial \mathcal{X}} \cong D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}})$$

Proof. The functor is pretty easy to construct. It is induced by the pullback functor

$$D^b \operatorname{Coh}^{G^\vee \times T^\vee}(\hat{\mathcal{N}}_{\mathcal{X}}) \to D^b \operatorname{Coh}^{G^\vee \times T^\vee}(\hat{\mathcal{N}})$$

along with an equivalence[8]

$$D^b \operatorname{Coh}^{G^\vee \times T^\vee}(\hat{\mathcal{N}}) \cong D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}}).$$

Verifying that the desired functor is an equivalence is mostly just an exercise in homological algebra and representation theory of algebraic groups. □

3.2. **Construction of the functor I.** We start by defining a functor

$$F_1 : \operatorname{Rep}(G^\vee \times T^\vee) \to P_I^{\text{Wak}}$$
$$V \otimes \Bbbk_{T^\vee}(\lambda) \mapsto Z(\operatorname{Sat}(V)) \star^I \mathcal{W}_\lambda$$

for $V \in \operatorname{Rep}(G^\vee)$ and $\lambda \in X^\vee$.[9] This functor is well-defined by Lemma 2.5.4 and Theorem 2.5.3.

The following is an easy consequence of Theorem 2.4.1 (3) and Proposition 2.5.2.

Lemma 3.2.1. *The functor F_1 is monoidal.*

3.3. **Construction of the functor II.** Next, we wish to extend the functor F_1 to a monoidal functor

$$F_2 : \operatorname{Coh}^{G^\vee \times T^\vee}_{\text{free}}(\hat{\mathcal{N}}_{\mathcal{X}}) \to P_I^{\text{Wak}}.$$

First, we replace P_I^{Wak} be a (non-full) subcategory $\mathcal{C}$ which is symmetric monoidal.[10] One can then check that F_1 factors through $\mathcal{C}$ to produce a functor

$$F_1' : \operatorname{Rep}(G^\vee \times T^\vee) \to \mathcal{C}.$$

By a Tannakian-like formalism, there is an equivalence

$$H : \mathcal{C} \to A\text{-mod}^{G^\vee \times T^\vee}_{\text{free}}$$

for some algebra A endowed a $G^\vee \times T^\vee$-action. It can be checked that via H, F_1' identifies with the functor

$$\operatorname{Rep}(G^\vee \times T^\vee) \to A\text{-mod}^{G^\vee \times T^\vee}_{\text{free}}$$
$$V \mapsto V \otimes A.$$

Extending F_1' to a functor $\operatorname{Coh}^{G^\vee \times T^\vee}_{\text{free}}(\hat{\mathcal{N}}_{\mathcal{X}}) \to P_I^{\text{Wak}}$ is then equivalent to a morphism $G^\vee \times T^\vee$-equivariant algebras

$$\mathcal{O}(\hat{\mathcal{N}}_{\mathcal{X}}) \to A.$$

This done by viewing $\mathcal{O}(\hat{\mathcal{N}}_{\mathcal{X}})$ as a quotient of $\mathcal{O}(\mathfrak{g}^\vee) \otimes \mathcal{O}(\mathcal{X})$. We start by defining equivariant algebra morphisms $\mathcal{O}(\mathfrak{g}^\vee) \to A$ and $\mathcal{O}(\mathcal{X}) \to A$. We give some brief remarks on how these are constructed:

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[8] This equivalence can be seen as just an isomorphism of underlying stacks $\hat{\mathcal{N}}/(G^\vee \times T^\vee) \cong \tilde{\mathcal{N}}/G^\vee$.

[9] It is not clear that this is functorial due to our presentation of Wakimoto sheaves. More precisely, one can define a functor $\operatorname{Rep}(T^\vee) \to P_I$ which takes $\Bbbk_{T^\vee}(\lambda)$ to $\mathcal{W}_\lambda$.

[10] This is somewhat necessitated by the observation that $\operatorname{Coh}^{G^\vee \times T^\vee}_{\text{free}}(\hat{\mathcal{N}}_{\mathcal{X}})$ is symmetric monoidal whereas P_I^{Wak} is not.

(1) The morphism $\mathcal{O}(\mathfrak{g}^\vee) \to A$ is effectively the logarithm of monodromy endomorphism for nearby cycles.
(2) The morphism $\mathcal{O}(\mathcal{X}) \to A$ is determined by the Wakimoto filtration for central sheaves given by Theorem 2.5.3.

It is then easy to check that the morphism $\mathcal{O}(\mathfrak{g}^\vee) \otimes \mathcal{O}(\mathcal{X}) \to A$ factors uniquely through the desired morphism $\mathcal{O}(\hat{\mathcal{N}}_{\mathcal{X}}) \to A$. The functor F_2 is then the composition of functors

$$F_2 : \mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}}) \to A\text{-mod}^{G^\vee \times T^\vee}_{\mathrm{free}} \cong \mathcal{C} \hookrightarrow P_I^{\mathrm{Wak}}.$$
$$M \mapsto M \otimes_{\mathcal{O}(\hat{\mathcal{N}}_{\mathcal{X}})} A.$$

3.4. **Construction of the functor III.** Recall from Proposition 3.1.2, that there is an equivalence of categories

$$K^b \,\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}}) / K^b \,\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}})_{\partial \mathcal{X}} \cong D^b \,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})$$

To check that F_2 extends to a functor

$$D^b \,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \to D^b P_I^{\mathrm{Wak}}$$

it then suffices to check that $K^b F_2$ takes $K^b \,\mathrm{Coh}^{G^\vee \times T^\vee}_{\mathrm{free}}(\hat{\mathcal{N}}_{\mathcal{X}})_{\partial \mathcal{X}}$ to acyclic complexes in $K^b P_I^{\mathrm{Wak}}$. This can be checked rather explicitly, and is omitted.

The upshot is there exist a functor

$$F : D^b \,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \to D^b P_I^{\mathrm{Wak}} \hookrightarrow D^b P_I,$$
$$V \otimes \mathcal{O}_{\tilde{\mathcal{N}}} \mapsto Z(\mathrm{Sat}(V)),$$
$$\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \mapsto \mathcal{W}_\lambda.$$

We will also refer to a variation of this functor F by composing with realization functor,

$$\overline{F} : D^b \,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \to D^b P_I \to D^b(I \backslash \mathrm{Fl}_G).$$

We emphasize that the realization functor for I-equivariant perverse sheaves on Fl_G is not an equivalence.

4. Antispherical Category and Iwahori-Whittaker Sheaves

We will construct another categorification of the antispherical module. In particular, we will define the category of *Iwahori-Whittaker perverse sheaves*, denoted $P_{\mathcal{IW}}$ along with an exact, fully faithful functor

$$P_I^{\mathrm{asph}} \to P_{\mathcal{IW}}.$$

We will show that the image of the central sheaves under the functor

$$\mathrm{Rep}(G^\vee) \to P_I \to P_I^{\mathrm{asph}} \to P_{\mathcal{IW}}$$

are the *tilting objects*.

Remark 4.0.1. *In order to make sense of Iwahori-Whittaker sheaves, we will need to depart from working in the Betti setting.*[11] *Instead, we will fix $\mathbb{F}$ to be an algebraic closure of $\mathbb{F}_p$. Now G will be a connected reductive group over $\mathbb{F}$, and likewise all ind-schemes will be over $\mathbb{F}$. We will also fix $\ell \neq p$, and all of our constructible*

[11] This is not strictly necessary. We could work with the Krillov model ([Gai2]) in the Betti setting, but such considerations are beyond the scope of this note.

sheaves should be interpreted as étale $\overline{\mathbb{Q}}_\ell$-sheaves. Everything discussed so far has an obvious analogue for étale sheaves.

4.1. **Warmup: Whittaker sheaves.** First, we will explain a simplistic version of Whittaker sheaves. Let X be an H variety where H is an algebraic group (not necessarily reductive). Let $a : H \times X \to X$ denote the action map. Fix a homomorphism $\mathcal{X} : H \to \mathbb{G}_m$. One can consider the Artin-Schreier local system $\mathcal{L}_{\mathrm{AS}}$ on $\mathbb{G}_a$.[12] The pullback $\mathcal{X}^*\mathcal{L}_{\mathrm{AS}}$ is a multiplicative local system on H. One can then consider the $(H, \mathcal{X}^*\mathcal{L}_{\mathrm{AS}})$-equivariant derived category of constructible sheaves on X, denoted $D^b_{\mathrm{c}}(H\backslash_{\mathcal{X}^*\mathcal{L}_{\mathrm{AS}}} X)$.

Intuitively, but not precisely[13], objects of $D^b_{\mathrm{c}}(H\backslash_{\mathcal{X}^*\mathcal{L}_{\mathrm{AS}}} X)$ consist of sheaves $\mathcal{F} \in D^b_{\mathrm{c}}(X)$ along with a natural isomorphism

$$a^*\mathcal{F} \cong \mathcal{X}^*\mathcal{L}_{\mathrm{AS}} \boxtimes \mathcal{F}.$$

Whittaker sheaves tend to satisfy strong cohomological vanishing properties motivated by the following essential property:

$$H^\bullet(\mathbb{G}_a, \mathcal{L}_{\mathrm{AS}}) \cong H^\bullet_c(\mathbb{G}_a, \mathcal{L}_{\mathrm{AS}}) = 0.$$

As a result, Whittaker sheaves are often tightly controlled:

Example 4.1.1 Let $\mathbb{G}_a$ act on $\mathbb{P}^1$ via $t \cdot [x : y] = [x + yt : y]$. Then there are two $\mathbb{G}_a$-orbits on $\mathbb{P}^1$– one isomorphic to $\mathbb{A}^1$ and the other isomorphic to a point. The point strata is too small to admit a Whittaker equivariant sheaf since any sheaf on a point is $\mathbb{G}_a$-equivariant. As a result,

$$D^b_{\mathrm{c}}(\mathbb{G}_a\backslash_\chi \mathbb{P}^1) \cong D^b_{\mathrm{c}}(\mathbb{G}_a\backslash_\chi \mathbb{A}^1).$$

It is then easy to check that $D^b_{\mathrm{c}}(\mathbb{G}_a\backslash_\chi \mathbb{A}^1) \cong D^b(\overline{\mathbb{Q}}_\ell\text{-mod})$ induced by taking the stalk at $0 \in \mathbb{A}^1$.

4.2. **The Iwahori-Whittaker category.** Let I_u^+ be the pro-unipotent radical if I^+, i.e., the preimage of U^+ under the natural map $LG \to G$. We will fix pinnings of U^+ to obtain an isomorphism $U^+/[U^+, U^+] \cong \prod \mathbb{G}_a$. Define a group homomorphism obtained as the composition

$$\mathcal{X} : I_u^+ \to U^+ \to U^+/[U^+, U^+] \cong \prod \mathbb{G}_a \overset{\mathrm{sum}}{\to} \mathbb{G}_a$$

We define the category of *Iwahori-Whittaker sheaves*, denoted $D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$, as the $(I_U^+, \mathcal{X}^*(\mathcal{L}_{\mathrm{AS}}))$-equivariant derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on Fl_G. The perverse t-structure on $D^b_{\mathrm{c}}(\mathrm{Fl}_G)$ restricts to that on $D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$. We denote the category of perverse Iwahori-Whittaker sheaves by $P_{\mathcal{IW}}$. There is a right action of $D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G)$ on $D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$ defined similarly to the convolution on $D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G)$. Namely, there is a bifunctor

$$(-) \star^I (-) : D^b_{\mathcal{IW}}(\mathrm{Fl}_G) \times D^b_{\mathrm{c}}(I\backslash \mathrm{Fl}_G) \to D^b_{\mathcal{IW}}(\mathrm{Fl}_G).$$

[12] Concretely, we choose a primitive p-th root of unity $\zeta \in \overline{\mathbb{Q}}_\ell$. There is a direct summand of the pushforward of the constant sheaf under the Galois covering $\mathbb{G}_a \to \mathbb{G}_a$ defined by $x \mapsto x^p - x$ on which $\mathbb{Z}/p\mathbb{Z}$ acts via the character $[n] \mapsto \zeta^n$

[13] To make this precise, one must incorporate higher coherence relations given by the multiplication maps $H^k \times X \to X$, c.f. [Gai2]

4.3. **Highest Weight Structure.** The I_u^+-orbits on Fl_G are parameterized by W_{ext}. There is a bijection

$$X^\vee \leftrightarrow {}^f W_{\mathrm{ext}}$$
$$\lambda \mapsto w_\lambda$$

Let $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}$ denote the I_u^+-orbit corresponding to w_λ, and let $j_\lambda^{\mathcal{IW}} : \mathrm{Fl}_{G,\lambda}^{\mathcal{IW}} \hookrightarrow \mathrm{Fl}_G$ denote the embedding.

The only I_u^+-orbits on Fl_G which admit nonzero Whittaker local systems are those indexed by $w \in {}^f W_{\mathrm{ext}}$. In fact, for each $\lambda \in X^\vee$, there is a unique rank-1 Whittaker local system, denoted $\mathcal{L}_{\mathcal{X},\lambda}$ on $\mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}$. For each $\lambda \in X^\vee$ we define sheaves

$$\Delta_\lambda^{\mathcal{IW}} = j_{\lambda !}^{\mathcal{IW}} \mathcal{L}_{\mathcal{X},\lambda}[\dim \mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}], \qquad \nabla_\lambda = j_{\lambda *}^{\mathcal{IW}} \mathcal{L}_{\mathcal{X},\lambda}[\dim \mathrm{Fl}_{G,\lambda}^{\mathcal{IW}}].$$

These sheaves are called the *standards* and *costandard* sheaves, respectively. They are naturally Iwahori-Whittaker perverse sheaves. The image of the canonical morphism $\Delta_\lambda^{\mathcal{IW}} \to \nabla_\lambda^{\mathcal{IW}}$ is a simple Iwahori-Whittaker perverse sheaf, denoted $\mathrm{IC}_\lambda^{\mathcal{IW}}$.

Proposition 4.3.1. *The category $P_{\mathcal{IW}}$ has a natural structure of a highest weight category with weight poset $X^\vee$.*

The key observation of Proposition 4.3.1 is that $P_{\mathcal{IW}}$ admits tilting sheaves. Namely, for each $\lambda \in X^\vee$, there is a unique indecomposable Iwahori-Whittaker tilting perverse sheaf $\mathcal{T}_\lambda^{\mathcal{IW}}$.

Corollary 4.3.2. *The realization functor induces an equivalence of categories*

$$D^b P_{\mathcal{IW}} \cong D^b_{\mathcal{IW}}(\mathrm{Fl}_G).$$

4.4. **Iwahori-Whittaker averaging.** Define the *Iwahori-Whittaker averaging functor*

$$\mathrm{Av}_{\mathcal{IW}} : D^b(I \backslash \mathrm{Fl}_G) \to D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$$
$$\mathcal{F} \mapsto \Delta_0^{\mathcal{IW}} \star^I \mathcal{F}.$$

Theorem 4.4.1 (Arkhipov-Bezrukavnikov, [ArBe])**.** *(1) The functor $\mathrm{Av}_{\mathcal{IW}}$ is perverse t-exact.*
(2) There restriction of $\mathrm{Av}_{\mathcal{IW}}$ to perverse sheaves, $P_I \to P_{\mathcal{IW}}$ factors through a fully faithful functor $P_I^{\mathrm{asph}} \to P_{\mathcal{IW}}$.

Proof Sketch of Theorem 4.4.1. By general geometric considerations and looking at orbits, one can show that

$$\mathrm{Av}_{\mathcal{IW}}(\mathrm{IC}_w) = 0 \tag{5}$$

unless $w \in {}^f W_{\mathrm{ext}}$. Moreover, for any $w \in W_{\mathrm{ext}}$, there are isomorphisms

$$\mathrm{Av}_{\mathcal{IW}}(\Delta_w) \cong \Delta_\lambda^{\mathcal{IW}}, \qquad \mathrm{Av}_{\mathcal{IW}}(\nabla_w) \cong \nabla_\lambda^{\mathcal{IW}} \tag{6}$$

where $\lambda \in X^\vee$ is the unique element such that $W \cdot w = W \cdot w_\lambda$.

A basic homological algebra allows one to deduce from 5 and 6 that $\mathrm{Av}_{\mathcal{IW}}$ is perverse t-exact. Moreover, from 5, $\mathrm{Av}_{\mathcal{IW}}$ will factor through the aspherical Hecke category.

In order to prove that the induced functor $\mathrm{Av}_{\mathcal{IW}} : P_I^{\mathrm{asph}} \to P_{\mathcal{IW}}$ is fully faithful, one constructs a section of $\mathrm{Av}_{\mathcal{IW}}$. Intuitively, this is done by averaging from $I \cap I_u^+$-equivariance to I-equivariance, taking perverse cohomology, and then applying the quotient functor to the aspherical Hecke category. □

4.5. **Central sheaves and Iwahori-Whittaker tilting sheaves.** The main result and motivation for introducing Iwahori-Whittaker sheaves comes from the following theorem.

Theorem 4.5.1 (Arkhipov-Bezrukavnikov, [ArBe])**.** *Consider the composition of functors*

$$Z^{\mathcal{IW}} : \mathrm{Rep}(G^\vee) \overset{Z}{\to} P_I \overset{\mathrm{Av}_{\mathcal{IW}}}{\to} P_{\mathcal{IW}}$$

For all $V \in \mathrm{Rep}(G^\vee)$, the perverse sheaf $Z^{\mathcal{IW}}(V)$ is tilting. Moreover, for any $\lambda \in X^\vee$,

$$(Z^{\mathcal{IW}}(V) : \Delta_\lambda^{\mathcal{IW}}) = (Z^{\mathcal{IW}}(V) : \nabla_\lambda^{\mathcal{IW}}) = \dim V_\lambda.$$

Proof Sketch of Theorem 4.5.1. The basic idea of the proof is to reduce until the remaining cases can be checked explicitly by hand. An outline of steps is given below.

(1) If $V, V' \in \mathrm{Rep}(G^\vee)$ such that $Z^{\mathcal{IW}}(V)$ and $Z^{\mathcal{IW}}(V')$ are tilting, then $Z^{\mathcal{IW}}(V \otimes V')$ is tilting. This basically follows from properties of convolution in $D^b(I\backslash \mathrm{Fl}_G)$.
(2) If the theorem holds G semisimple group, then it holds for G reductive.
(3) By (1) and (2), it suffices to assume that G is semisimple and V is a simple $G^\vee$-module whose highest weight is either miniscule or quasi-miniscule. The miniscule case is rather strightforward computation. Whereas the quasi-miniscule case is much more difficult and requires studying certain quotients of P_I.

□

5. The Equivalence

We consider the functor

$$F_{\mathcal{IW}} : D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \to D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$$

defined as the composition

$$D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \overset{F}{\to} D^b P_I \to D^b_c(I\backslash \mathrm{Fl}_G) \overset{\mathrm{Av}_{\mathcal{IW}}}{\to} D^b_{\mathcal{IW}}(\mathrm{Fl}_G).$$

Alternatively, by Theorem 4.4.1 and Corollary 4.3.2 it can be defined as the composition

$$D^b \operatorname{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \overset{F}{\to} D^b P_I \overset{\mathrm{Av}_{\mathcal{IW}}}{\to} D^b P_{\mathcal{IW}} \cong D^b_{\mathcal{IW}}(\mathrm{Fl}_G).$$

We can check that $F_{\mathcal{IW}}$ maps objects as follows:

$$\begin{array}{ccccc} D^b \operatorname{Coh}(\tilde{\mathcal{N}}/G^\vee) & \xrightarrow{\overline{F}} & D^b(I\backslash \mathrm{Fl}_G) & \xrightarrow{\mathrm{Av}_{\mathcal{IW}}} & D^b_{\mathcal{IW}}(\mathrm{Fl}_G) \\ V \otimes \mathcal{O}_{\tilde{\mathcal{N}}} & \longmapsto & Z(\mathrm{Sat}(L_\lambda)) & \longmapsto & \mathcal{T}^{\mathcal{IW}}(V) \\ \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) & \longmapsto & \mathcal{W}_\lambda & \longmapsto & \mathrm{Av}_{\mathcal{IW}}(\mathcal{W}_\lambda) \overset{K_0}{\sim} \Delta_\lambda^{\mathcal{IW}} \end{array} \tag{7}$$

Note that the fact that $F_{\mathcal{IW}}(L_\lambda \otimes \mathcal{O}_{\tilde{\mathcal{N}}})$ is tilting follows from Theorem 4.5.1. We write $\mathcal{T}^{\mathcal{IW}}(V)$ for this tilting object.[14]

Additionally, $\mathrm{Av}_{\mathcal{IW}}(\mathcal{W}_\lambda)$ is not isomorphic to the standard object $\Delta_\lambda^{\mathcal{IW}}$, but they represent the same class in the Grothendieck group of $D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$.

We finally arrive at the main theorem– the Arkhipov-Bezrukavnikov equivalence.

Theorem 5.0.1 (Arkhipov-Bezrukavnikov, [ArBe])**.** *There functor $F_{\mathcal{IW}} : D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \to D^b_{\mathcal{IW}}(\mathrm{Fl}_G)$ is an equivalence of categories.*

Proof Sketch of Theorem 5.0.1. The fact that $F_{\mathcal{IW}}$ is essentially surjective follows from the description given in (7) along with a routine homological algebra argument.

In order to show that for $\mathcal{F}, \mathcal{G} \in D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})$, the morphism

$$\mathrm{Hom}_{D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{D^b_{\mathcal{IW}}(\mathrm{Fl}_G)}(F_{\mathcal{IW}}(\mathcal{F}), F_{\mathcal{IW}}(\mathcal{G})) \tag{8}$$

is an isomorphism, it suffices by a 5-lemma argument to consider the case of $\mathcal{F} = \mathcal{O}_{\tilde{\mathcal{N}}}$ and $\mathcal{G} = V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)[n]$ for some $V \in \mathrm{Rep}(G^\vee)$, some $\lambda \in X_+^\vee$ and some $n \in \mathbb{Z}$. The fact that (8) is injective is mostly straightforward and representation theoretic. Moreover, it is not hard to check that

$$\dim \mathrm{Hom}_{D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})}(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)[n]) = \begin{cases} \dim V_{-\lambda} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Note that there are isomorphisms

$$F_{\mathcal{IW}}(\mathcal{O}_{\tilde{\mathcal{N}}}) = \Delta_0^{\mathcal{IW}}, \qquad F_{\mathcal{IW}}(V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \cong Z^{\mathcal{IW}}(V) \star^I \mathcal{W}_\lambda.$$

On the left-hand side of (8),

$$\begin{aligned} &\mathrm{Hom}_{D^b_{\mathcal{IW}}(\mathrm{Fl}_G)}(\Delta_0^{\mathcal{IW}}, Z^{\mathcal{IW}}(V) \star^I \mathcal{W}_\lambda[n]) \cong \mathrm{Hom}_{D^b_{\mathcal{IW}}(\mathrm{Fl}_G)}(\Delta_0^{\mathcal{IW}} \star^I \mathcal{W}_{-\lambda}, Z^{\mathcal{IW}}(V)[n]) && (2.5.2) \\ &\cong \mathrm{Hom}_{D^b_{\mathcal{IW}}(\mathrm{Fl}_G)}(\Delta_{-\lambda}^{\mathcal{IW}}, Z^{\mathcal{IW}}(V)[n]) && (6) \end{aligned}$$

We then conclude by Theorem 4.5.1 that

$$\dim \mathrm{Hom}_{D^b_{\mathcal{IW}}(\mathrm{Fl}_G)}(\Delta_0^{\mathcal{IW}}, Z^{\mathcal{IW}}(V) \star^I \mathcal{W}_\lambda[n]) = \begin{cases} \dim V_{-\lambda} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Therefore, $F_{\mathcal{IW}}$ is fully faithful. □

Corollary 5.0.2 (Arkhipov-Bezrukavnikov, [ArBe])**.** *There are equivalences of categories*

$$\begin{tikzcd} D^b\,\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}) \arrow[r, "\sim"] \arrow[dr] & D^b P_I^{\mathrm{asph}} \arrow[r, "\sim"] & D^b P_{\mathcal{IW}} \arrow[r, "\sim"] & D^b_{\mathcal{IW}}(\mathrm{Fl}_G) \\ & D^b P_I \arrow[u] \arrow[ur] & & \end{tikzcd}$$

[14] It is natural to ask whether $\mathcal{T}^{\mathcal{IW}}(L_\lambda) \cong \mathcal{T}_\lambda^{\mathcal{IW}}$ for $\lambda \in X_+^\vee$. This is in fact the case, and can be obtained as a corollary of Theorem 5.0.1.

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